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A short proof of Ramanujan's famous $_1\psi_1$ summation formula

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Abstract

We present a new proof of Ramanujan's $_1\psi_1$ summation formula.

Keywords: Special function; Ramanujan; $_1\psi_1$

1. Introduction

The celebrated $_1\psi_1$ summation theorem was first recorded by Ramanujan in his second notebook [24] in approximately 1911–1913. However, because his notebooks were not published until 1957, it was not brought before the mathematical public until 1940 when G.H. Hardy recorded Ramanujan's $_1\psi_1$ summation theorem in his treatise on Ramanujan's work [17, pp. 222–223]. Subsequently, the first published proofs were given in 1949 and 1950 by Hahn [16] and Jackson [20], respectively. Since these first two proofs, several others have been published, namely, by Andrews [2,3], Ismail [18], Andrews and Askey [4], Askey [6], Adiga, et al. [1], Kadell [21], Fine [14, Eq. (18.3)], Mimachi [23], Corteel and Lovejoy [13], Corteel [12], and Yee [26]. The proof given in [1] and reproduced in [7, Entry 17, p. 32] was, in fact, first given in lectures at the University of Mysore by K. Venkatachaliengar in the 1960s. The proofs by Corteel and Yee are combinatorial. In particular, Yee devised a

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bijection between the partitions generated on each side of the ${}_1\psi_1$ identity. There is also an unpublished proof by Liu [22].

In this paper, we use partial fractions to give a new, short proof of Ramanujan's ${}_1\psi_1$ summation theorem. Watson [25] utilized partial fractions to prove some of Ramanujan's theorems on mock theta functions. In the past few years, it has become increasingly apparent that Ramanujan employed partial fractions in proving theorems in the theory of q-series, in particular, about mock theta functions, and in other areas of analysis as well; see, e.g., [5, Chapter 12], [8]. This paper is motivated by the work in [5, Chapter 12] and [19] and continues the work in [10]. Of course, we have no evidence that Ramanujan found the proof that we give here. We record Ramanujan's ${}_1\psi_1$ summation ([7, Entry 17, p. 32]) in the form recorded in his second notebook.

Theorem 1.1. *Suppose that* $|\beta q| < |z| < 1/|\alpha q|$. *Then*

$$1+\sum_{k=1}^{\infty}\frac{(1/\alpha;q^2)_k(-\alpha q)^k}{(\beta q^2;q^2)_k}z^k+\sum_{k=1}^{\infty}\frac{(1/\beta;q^2)_k(-\beta q)^k}{(\alpha q^2;q^2)_k}z^{-k}$$
$$=\left\{\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty}\right\}\left\{\frac{(\alpha\beta q^2;q^2)_\infty(q^2;q^2)_\infty}{(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}\right\}. \tag{1.1}$$

In (1.1) and the remainder of the paper, we use the customary notation $(x;q)_0 := 1$, $(x;q)_n := \prod_{k=0}^{n-1}\left(1-xq^k\right)$ for $n \geqslant 1$, and $(x;q)_\infty := \prod_{k=0}^{\infty}(1-xq^k)$. Here and throughout this article, we assume that $|q| < 1$.

We present our proof of Theorem 1.1 in the next section.

2. Proof

We require the q-binomial theorem [7, Entry 2, p. 14], [14, Eq. (6.2)] in our proof.

Lemma 2.1 (*The q-binomial theorem.*). *If* $|q|, |a| < 1$, *then*

$$\frac{(at;q)_\infty}{(a;q)_\infty}=\sum_{n=0}^{\infty}\frac{a^n(t;q)_n}{(q;q)_n}.$$

Proof of Theorem 1.1. Applying the Mittag–Leffler theorem, we consider the partial fraction decomposition of the finite product

$$\frac{(-qz;q^2)_N(-q/z;q^2)_N}{(-\alpha qz;q^2)_{N+1}(-\beta q/z;q^2)_{N+1}}=\sum_{n=0}^{N}\left\{\frac{A(n)}{1+\alpha q^{2n+1}z}+\frac{B(n)}{z+\beta q^{2n+1}}\right\}+F_N(q;z), \tag{2.1}$$

where $F_N(q;z)$ is an entire function of z.

Note that the left side of (2.1) tends to 0 as $z \to \infty$, which can be seen immediately by multiplying both the numerator and the denominator by z^{N+1}. The finite sum on the right side also tends to 0 as $z \to \infty$. Therefore F_N also tends to 0 as $z \to \infty$. Hence F_N is a bounded entire function and must be equal to 0, since F_N tends to 0 as $z \to \infty$. Since all the poles are simple, the values of $A(n)$ and $B(n)$ are calculated by, for $n \geqslant 0$,

$$A(n) = \lim_{z \to \frac{1}{-\alpha q^{2n+1}}} (1+\alpha q^{2n+1} z) \frac{(-qz;q^2)_N(-q/z;q^2)_N}{(-\alpha qz;q^2)_{N+1}(-\beta q/z;q^2)_{N+1}} = \frac{(q^{-2n}/\alpha;q^2)_n(1/\alpha;q^2)_{N-n}(\alpha q^{2n+2};q^2)_N}{(q^{-2n};q^2)_n(q^2;q^2)_{N-n}(\alpha\beta q^{2n+2};q^2)_{N+1}}, \tag{2.2}$$

while

$$B(n) = \lim_{z \to -\beta q^{2n+1}} (z+\beta q^{2n+1}) \frac{(-qz;q^2)_N(-q/z;q^2)_N}{(-\alpha qz;q^2)_{N+1}(-\beta q/z;q^2)_{N+1}} = \frac{-\beta q^{2n+1}(\beta q^{2n+2};q^2)_N(q^{-2n}/\beta;q^2)_n(1/\beta;q^2)_{N-n}}{(\alpha\beta q^{2n+2};q^2)_{N+1}(q^{-2n};q^2)_n(q^2;q^2)_{N-n}}. \tag{2.3}$$

By applying the elementary identity

$$(xq^{-n};q)_n = (-1)^n x^n q^{-n(n+1)/2}(q/x;q)_n,$$

we further simplify (2.2) and (2.3) to find that

$$A(n) = \frac{(1/\alpha)^n(\alpha\beta q^2;q^2)_n(1/\alpha;q^2)_{N-n}(\alpha q^2;q^2)_{N+n}}{(q^2;q^2)_n(q^2;q^2)_{N-n}(\alpha\beta q^2;q^2)_{N+n+1}} \tag{2.4}$$

and

$$B(n) = \frac{-\beta q^{2n+1}(1/\beta)^n(\alpha\beta q^2;q^2)_n(1/\beta;q^2)_{N-n}(\beta q^2;q^2)_{N+n}}{(q^2;q^2)_n(q^2;q^2)_{N-n}(\alpha\beta q^2;q^2)_{N+n+1}}. \tag{2.5}$$

Assuming that $1/|\alpha| < 1$ and $|q^2/\beta| < 1$, substituting (2.2) and (2.3) into (2.1), letting $N \to \infty$, and applying Tannery's theorem [9, Section 49], [15, Section 63] to justify letting $N \to \infty$ under the summation sign, we obtain

$$\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty} = \frac{(1/\alpha;q^2)_\infty(\alpha q^2;q^2)_\infty}{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1/\alpha)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(1+\alpha q^{2n+1}z)} - \frac{1}{z}\frac{(1/\beta;q^2)_\infty(\beta q^2;q^2)_\infty}{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{\beta q^{2n+1}(1/\beta)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(1+\beta q^{2n+1}/z)}.$$

Multiplying both sides by $\frac{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}$, we arrive at

$$\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty} = \frac{(1/\alpha;q^2)_\infty}{(\beta q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1/\alpha)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(1+\alpha q^{2n+1}z)}$$

$$
\begin{aligned}
&-\frac{(1/\beta;q^2)_\infty}{(\alpha q^2;q^2)_\infty}\frac{1}{z}\sum_{n=0}^{\infty}\frac{\beta q^{2n+1}(1/\beta)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(1+\beta q^{2n+1}/z)}\\
&=\frac{(1/\alpha;q^2)_\infty}{(\beta q^2;q^2)_\infty}\sum_{m=0}^{\infty}z^m\sum_{n=0}^{\infty}\frac{(1/\alpha)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n}(-\alpha q^{2n+1})^m\\
&\quad+\frac{(1/\beta;q^2)_\infty}{(\alpha q^2;q^2)_\infty}\sum_{m=1}^{\infty}z^{-m}\sum_{n=0}^{\infty}\frac{(1/\beta)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n}(-\beta q^{2n+1})^m. \qquad (2.6)
\end{aligned}
$$

Note that we have assumed further that $|\beta q| < |z| < 1/|\alpha q|$ so that the geometric expansions in the last equality are valid and the inversions of the order of summations in the last equality are valid by absolute convergence of the series.

Comparing the coefficients of z^m in (1.1) and (2.6), it suffices to show that

$$\frac{(1/\alpha;q^2)_m(-\alpha q)^m}{(\beta q^2;q^2)_m}=\frac{(1/\alpha;q^2)_\infty}{(\beta q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1/\alpha)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n}(-\alpha q^{2n+1})^m$$

and

$$\frac{(1/\beta;q^2)_m(-\beta q)^m}{(\alpha q^2;q^2)_m}=\frac{(1/\beta;q^2)_\infty}{(\alpha q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1/\beta)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n}(-\beta q^{2n+1})^m,$$

that is,

$$\frac{(\beta q^{2m+2};q^2)_\infty}{(q^{2m}/\alpha;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{2m}/\alpha)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n}$$

and

$$\frac{(\alpha q^{2m+2};q^2)_\infty}{(q^{2m}/\beta;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{2m}/\beta)^n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n},$$

and these are true by applying Lemma 2.1 with $(a,t)=(q^{2m}/\alpha,\alpha\beta q^2)$ and $(a,t)=(q^{2m}/\beta,\alpha\beta q^2)$, respectively. Therefore we have proved (1.1) for the region $|q^3|<|\beta q|<|z|<1/|\alpha q|<1/|q|$. By applying analytic continuation, we complete the proof. □

In a future paper, we shall use a similar method to prove the following lemma, which yields the q-binomial theorem as a special case. Details will be given in [11].

Lemma 2.2. *For $|\gamma/\alpha|<1$ and $|q^2/\beta|<1$, we have*

$$
\begin{aligned}
&\frac{(-\gamma qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty(\alpha q^2;q^2)_\infty(\beta\gamma q^2;q^2)_\infty}\\
&=\frac{(\gamma/\alpha;q^2)_\infty}{(\beta\gamma q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma/\alpha)^n(\alpha q^2/\gamma;q^2)_n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(\alpha q^2;q^2)_n(1+\alpha q^{2n+1}z)}\\
&\quad-\frac{1}{z}\frac{(1/\beta;q^2)_\infty}{(\alpha q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{\beta q^{2n+1}(1/\beta)^n(\beta q^2;q^2)_n(\alpha\beta q^2;q^2)_n}{(q^2;q^2)_n(\beta\gamma q^2;q^2)_n(1+\beta q^{2n+1}/z)}.
\end{aligned}
$$

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